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ON LOCALLY SYMMETRIC VECTOR FIELDS ON RIEMANNIAN MANIFOLDS

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ABSTRACT

It is shown that if an n-dimensional ($n \geqq 3$) Riemannian manifold admits $r \geqq 2$ locally symmetric vector fields (LSVF's), then it is a $V(k)$-space. In particular, if $r = n - 1$ then the manifold is a space of constant curvature. In the case of a 3-dimensional Riemannian manifold a close connection between LSVF's and eigenvectors of the Ricci tensor is found.

1. Introduction

This paper is concerned with n-dimensional ($n > 2$) Riemannian manifolds admitting $r \geqq 2$ linearly independent locally symmetric vector fields (briefly LSVF). LSVF's of the first and of the second order were defined in [6] by A. G. Walker and the definition was motivated by his investigation of possible laws of orientation of galaxies in the standard cosmological model of General Relativity ([5]).

In the present paper we show that the existence of several LSVF's imposes very strong restrictions on a Riemannian manifold. It turns out that for $n \geqq 4$ a Riemannian manifold admits $r \geqq 3$ linearly independent LSVF's of the first order or admits two such fields at least one of which is of the second order, if and only if it is a Riemannian manifold of a very special type ($V(k)$-space). It will be also shown that if for $n \geqq 4$ a Riemannian manifold admits $(n-1)$ linearly independent LSVF's of the first order, then it is a space of constant curvature.

The case $n = 3$ is special. It will be shown that in this case there is a close connection between LSVF's and eigenvectors of the Ricci tensor. It turns out also that if a 3-dimensional Riemannian manifold admits three different (but not necessarily linearly independent) LSVF's of the first order or two such fields one

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of which is of the second order, then this Riemannian manifold is a space of constant curvature.

2. Preliminaries

Let M be an n-dimensional Riemannian manifold ($n \geqq 2$) and let V be a unit vector field on M. Let U be a normal coordinate neighbourhood with the center at $p \in M$. According to [6], the field V is said to be *symmetric about* p if its restriction on U is invariant under all orthogonal transformations of the normal coordinates on U which leave $V(p)$ invariant. Let $(x) = (x^1, \cdots, x^n)$ be a normal coordinate system on U with the center at p and let $F(x) = 0$ be the condition on $V = V(x)$ for symmetry about p. In this case $F(0) = 0$ identically, and V is said to have *first order local symmetry about* p if $\partial F / \partial x^i = 0$ at p ($i = 1, \cdots, n$), and *second order local symmetry about* p if also $\partial^2 F / \partial x^i \partial x^j = 0$ at p ($i, j = 1, \cdots, n$). The vector field is said to be a *locally symmetric vector field* (*LSVF*) *of the first* (*second*) *order*, if it has first (second) order local symmetry about every point of M.

Proposition 1 ([6]). *Let M be an n-dimensional Riemannian manifold.*

(a) *For $n \geqq 4$ a unit vector field V on M is a LSVF of the first order if and only if there exists a function λ on M such that for every $X \in TM$*

$$\nabla_X V = \lambda(X - \langle X, V \rangle V), \tag{1}$$

where TM is the tangent bundle of M and $\langle \cdots, \cdots \rangle$ is the Riemannian scalar product.

This field is a LSVF of the second order if and only if in addition to (1) *it satisfies*

$$X(\lambda) = \langle X, V \rangle V(\lambda) \tag{2}$$

for every $X \in TM$.

(b) *For $n = 3$ a unit vector field V on M is a LSVF of the first order if and only if there exist two functions λ and β on M such that for every $X \in TM$*

$$\nabla_X V = \lambda(X - \langle X, V \rangle V) + \beta(V \times X), \tag{3}$$

where $V \times X$ is the "cross-product" in 3-dimensional Euclidean space.

This field is a LSVF of the second order if and only if in addition to (3) *it satisfies one of the following two conditions:*

(i) $$X(\lambda) = \langle X, V \rangle V(\lambda),$$

(4) $$\beta = 0$$

for every $X \in TM$.

(ii) $$\lambda = 0,$$

(5) $$\beta = \text{const} \neq 0.$$

In the following discussion we need the definition and some properties of $V(k)$-spaces. Such spaces were introduced and investigated by G. I. Kruckovic and A. S. Solodovnikov in [3], [4].

DEFINITION 1 ([1]). Given Riemannian manifolds M_0 and M_1 and a positive valued function f on M_0, the *warped product* $M = M_0 \times_f M_1$ is the manifold $M_0 \times M_1$ furnished with the Riemannian structure such that $\|X\|^2 = \|\pi_{0*}X\|^2 + f^2(\pi_0 x)\|\pi_{1*}X\|^2$ for every $X \in TM_x$, $x \in M$, where π_i $(i = 0, 1)$ is the projection $\pi_i : M_0 \times M_1 \to M_i$ and $\|\cdots\|$ is the norm on M_i.

DEFINITION 2 ([4]). A warped product $M = M_0 \times_f M_1$ is called a *k-decomposition* of M if $\dim M_0 \geqq 2$ and the manifold $M_0 \times_f R^1$ is a space of constant curvature k. M_0 is called the *principal part* of the k-decomposition.

PROPOSITION 2 ([4]). (a) *If a Riemannian manifold M admits a k-decomposition $M = M_0 \times_f M_1$ and an l-decomposition $M = N_0 \times_g N_1$, then $k = l$.*

(b) *If $M = M_0 \times_f M_1$ is a k-decomposition of M, then M_0 is a space of constant curvature k.*

(c) *Given a space M_0 of constant curvature k, a function ψ on M_0 and an arbitrary Riemannian manifold M_1, the warped product $M = M_0 \times_{e^{2\psi}} M_1$ is a k-decomposition of M if and only if ψ satisfies the condition*

(6) $$\nabla_X \operatorname{grad} \psi = -kX - X(\psi) \operatorname{grad} \psi,$$

where $X \in TM_0$ and ∇ is the covariant derivative on M_0.

DEFINITION 3 ([4]). A Riemannian manifold M is called a *$V(k)$-space* if for every $p \in M$ there exists a neighbourhood $U \ni p$ admitting a k-decomposition $U = U_0 \times_f U_1$.

DEFINITION 4. Given a $V(k)$-space M and $p \in M$, let Λ be a set of all neighbourhoods of p which admit a k-decomposition. The maximal value of $\dim U_0$ for all k-decompositions $U = U_0 \times_f U_1$, $U \in \Lambda$ is called the *range* of M at the point p.

It is convenient to regard an n-dimensional space of constant curvature k as a $V(k)$-space of the range n at every point.

3. The case $n \geqq 4$

In this section we investigate n-dimensional ($n \geqq 4$) Riemannian manifolds admitting several LSVF's. We will denote a LSVF V satisfying the equation (1) by (V, λ).

THEOREM 1. (a) *Let M be an n-dimensional ($n \geqq 4$) Riemannian manifold admitting either $r \geqq 3$ linearly independent LSVF's of the first order or $r = 2$ such fields at least one of which is of the second order. Then each of these fields is a LSVF of the second order.*

(b) *If an n-dimensional ($n \geqq 4$) Riemannian manifold M admits $r \geqq 2$ linearly independent LSVF's (V_i, λ_i) $(i = 1, \cdots, r)$ of the second order then it is a $V(k)$-space. For every point $p \in M$ there exist a neighbourhood $U \ni p$, a k-decomposition $U = U_0 \times_{e^{2\psi}} U_1$ with* $\dim U_0 = r$, *and r LSVF's (W_i, μ_i) of the second order on U_0 such that $W_i(\psi) = \mu_i$, $V_i(q) = \varphi_{q_1 *}(W_i(q_0))$, $\lambda_i(q) = \mu_i(q_0)$, where $q = (q_0, q_1)$ is an arbitrary point of $U = U_0 \times_{e^{2\psi}} U_1$ $\varphi_{q_1} : U_0 \to U$, $\varphi_{q_1}(q_0) = (q_0, q_1)$; and $\pi_0 : U \to U_1$ is the natural projection.*

(c) *If M_0 is a $V(k)$-space, $p \in M$, and the range of M at p is r, then there exist a neighbourhood $U \ni p$, a k-decomposition $U = U_0 \times_{e^{2\psi}} U_1$ with* $\dim U_0 = r$, *and r LSVF's (W_i, μ_i) $(i = 1, \cdots, r)$ of the second order on U_0 such that $W_i(\psi) = \mu_i$ and the vector fields (V_i, λ_i) on U arising from (W_i, μ_i) as prescribed in* (b), *are LSVF's of the second order on U.*

(d) *If M is a connected and simply connected r-dimensional Riemannian manifold of constant curvature k, and (W_i, μ_i) $(i = 1, \cdots, r)$ are LSVF's of the second order on M_0, then there exists a function ψ on M_0 satisfying the equations $W_i(\psi) = \mu_i$. If M_1 is an arbitrary Riemannian manifold, then $M = M_0 \times_{e^{2\psi}} M_1$ is a $V(k)$-space and the vector fields (V_i, λ_i) $(i = 1, \cdots, r)$ arising from (W_i, μ_i) as prescribed in* (b), *are LSVF's of the second order on M.*

(e) *If an n-dimensional ($n \geqq 4$) Riemannian manifold admits $(n-1)$ linearly independent LSVF's of the first order, then it is a space of constant curvature.*

I. PROOF OF (a). Let (V_i, λ_i) $(i = 1, \cdots, r; r \geqq 2)$ be LSVF's of the first order on M. Then by (1)

$$\nabla_X V_i = \lambda_i (X - \langle X, V_i \rangle V_i). \tag{7}$$

Let us denote $\tau_{ij} = \langle V_i, V_j \rangle$ and let $\chi(M)$ be the set of all vector fields on M.

LEMMA 1. *Let $X \in \chi(M)$. If for some i, j $(i \neq j)$ $\langle X, V_i \rangle = 0, \langle X, V_j \rangle = 0$, then*

$$\langle [X, V_i], V_i \rangle = 0, \quad \langle [X, V_i], V_j \rangle = 0, \quad X(\tau_{ij}) = 0, \quad X(\lambda_i) = 0.$$

PROOF. $V_i \langle X, V_j \rangle = 0$. Using (7) we obtain $\langle \nabla_{V_i} X, V_j \rangle = 0$. Again by (7), $\langle [X, V_i], V_j \rangle = \langle \nabla_X V_i - \nabla_{V_i} X, V_j \rangle = \lambda_i \langle X, V_j \rangle = 0$. It can be proved analogously that $\langle [X, V_i], V_i \rangle = 0$.

Also $X(\tau_{ij}) = X\langle V_i, V_j \rangle = \langle \nabla_X V_i, V_j \rangle + \langle V_i, \nabla_X V_j \rangle = 0$, by (7).

Since $\langle [X, V_j], V_i \rangle = 0$, $\langle [X, V_j], V_j \rangle = 0$, we obtain $[X, V_j](\tau_{ij}) = 0$. It follows that

$$0 = [X, V_j](\tau_{ij}) = X(V_j \tau_{ij}) - V_j(X \tau_{ij}) = X(V_j \langle V_i, V_j \rangle)$$
$$= X(\langle \nabla_{V_j} V_i, V_j \rangle + \langle V_i, \nabla_{V_j} V_j \rangle) = X(\lambda_i (1 - \tau_{ij}^2)) = X(\lambda_i)(1 - \tau_{ij}^2).$$

Therefore $X(\lambda_i) = 0$. ∎

Let us denote

$$\eta_{ij} = V_i(\lambda_j) + \lambda_j^2 \tau_{ij}. \tag{8}$$

LEMMA 2. (i) $\eta_{11} = \eta_{22} = \cdots = \eta_{rr}$.

(ii) $\eta_{ij} = \eta_{ji}$.

(iii) *For every $X, Y \in \chi(M)$*

$$\langle R(X, V_i) V_i, Y \rangle = k(\langle X, Y \rangle - \langle X, V_i \rangle \langle Y, V_i \rangle) \tag{9}$$

where $k = -\eta_{ii}$, i.e.,

$$k = -[V_i(\lambda_i) + \lambda_i^2]. \tag{10}$$

PROOF. Computing $(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}) V_i$ and using (7), we obtain

$$\begin{aligned} R(X, Y) V_i = & -[Y(\lambda_i) + \lambda_i^2 \langle Y, V_i \rangle] X + [X(\lambda_i) + \lambda_i^2 \langle X, V_i \rangle] Y \\ & + [Y(\lambda_i) \langle X, V_i \rangle - X(\lambda_i) \langle Y, V_i \rangle] V_i. \end{aligned} \tag{11}$$

It follows that $\langle R(V_j, V_i) V_i, V_j \rangle = -\eta_{ii}(1 - \tau_{ij}^2)$. Since

$$\langle R(V_j, V_i) V_i, V_j \rangle = \langle R(V_i, V_j) V_j, V_i \rangle,$$

we obtain $\eta_{ii} = \eta_{jj}$. This proves (1).

Using (11) we obtain $\langle R(X, V_i) V_i, Y \rangle = k(\langle X, Y \rangle - \langle X, V_i \rangle \langle Y, V_i \rangle)$, where k is defined by (10). This proves (iii).

It follows from (11) that

$$\langle R(X, V_j)V_i, X\rangle = -\eta_{ij}(\langle X, X\rangle - \langle X, V_i\rangle^2) + [X(\lambda_i) + \lambda_i^2\langle X, V_i\rangle]\langle X, V_j - \tau_{ij}V_i\rangle. \tag{12}$$

Suppose that $i \neq j$ and let $\{e_a\}$ $(a = 1, \cdots, n)$ be an orthonormal base such that $e_1 = V_i$, $e_2 = 1/\sqrt{1-\tau_{ij}^2}(V_j - \tau_{ij}V_i)$, $\langle e_\alpha, V_i\rangle = 0$, $\langle e_\alpha, V_j\rangle = 0$ $(\alpha = 3, \cdots, n)$. Using (12), (8), (10) and Lemma 1, we can calculate the Ricci tensor $S(V_j, V_i)$:

$$S(V_j, V_i) = \sum_{a=1}^{n} \langle R(e_a, V_j)V_i, e_a\rangle = -\eta_{ji}(n-2) + k\tau_{ij}.$$

Since $S(V_j, V_i) = S(V_i, V_j)$ and since $n \neq 2$, we obtain $\eta_{ij} = \eta_{ji}$. This proves (ii). ■

LEMMA 3. *If the conditions of statement* (a) *of Theorem* 1 *are satisfied, then*

$$\eta_{ij} = -k\tau_{ij}. \tag{13}$$

PROOF. First we consider the case $r \geqq 3$. Let i, j, k $(1 \leqq i, j, k \leqq r)$ be pairwise distinct. Let us denote

$$\tilde{V}_k = V_k - (1-\tau_{ij}^2)^{-1}[(\tau_{ik} - \tau_{jk}\tau_{ij})V_i + (\tau_{jk} - \tau_{ik}\tau_{ij})V_j]. \tag{14}$$

One can verify that $\langle \tilde{V}_k, V_i\rangle = 0$, $\langle \tilde{V}_k, V_j\rangle = 0$. By Lemma 1, $\tilde{V}_k(\lambda_i) = 0$, $\tilde{V}_k(\lambda_j) = 0$. Therefore by (14),

$$V_k(\lambda_i)(1-\tau_{ij}^2) = V_i(\lambda_i)(\tau_{ik} - \tau_{jk}\tau_{ij}) - V_j(\lambda_i)(\tau_{jk} - \tau_{ik}\tau_{ij});$$

$$V_k(\lambda_j)(1-\tau_{ij}^2) = V_j(\lambda_j)(\tau_{jk} - \tau_{ik}\tau_{ij}) - V_i(\lambda_j)(\tau_{ik} - \tau_{jk}\tau_{ij}).$$

Using (8), (10) and Lemma 2 (ii), we obtain

$$\eta_{ik}(1-\tau_{ij}^2) - \eta_{ij}(\tau_{jk} - \tau_{ik}\tau_{ij}) + k(\tau_{ik} - \tau_{jk}\tau_{ij}) = 0;$$

$$\eta_{jk}(1-\tau_{ij}^2) - \eta_{ij}(\tau_{ik} - \tau_{jk}\tau_{ij}) + k(\tau_{jk} - \tau_{ik}\tau_{ij}) = 0.$$

Taking the sum and the difference of these two equations, we infer

$$(\eta_{ik} + \eta_{jk})(1+\tau_{ij}) + (k - \eta_{ij})(\tau_{ik} + \tau_{jk}) = 0;$$

$$(\eta_{ik} + \eta_{jk})(1-\tau_{ij}) + (k + \eta_{ij})(\tau_{ik} + \tau_{jk}) = 0.$$

Taking the sum of these equations, we get

$$\eta_{ik} + \eta_{jk}\tau_{ij} - \eta_{ij}\tau_{jk} + k\tau_{ik} = 0.$$

Analogously we obtain

$$\eta_{ki} + \eta_{ji}\tau_{kj} - \eta_{kj}\tau_{ji} + k\tau_{ki} = 0.$$

Taking the sum of the last two equations, we find $\eta_{ik} = -k\tau_{ik}$.

Suppose now that there are only two LSVF's (V_1, λ_1) and (V_2, λ_2), and suppose that (V, λ_2) is a LSVF of the second order. By (2), this means that for every $X \in \chi(M)$, $X(\lambda_2) = \langle X, V_2\rangle V_2(\lambda_2)$. Substituting $X = V_1$ and using (8) and (10) we obtain $\eta_{12} = -k\tau_{12}$. This proves the Lemma. ∎

Now we are ready to prove statement (a) of Theorem 1. By (2), we have to prove that for every $X \in \chi(M)$

$$X(\lambda_i) = \langle X, V_i\rangle V_i(\lambda_i). \tag{15}$$

It is sufficient to consider the cases:

(i) $\langle X, V_i\rangle = 0$ $(i = 1, \cdots, r)$;

(ii) $X = V_i$;

(iii) $X = V_j$ $(j \neq i)$

In case (i) $X(\lambda_i) = 0$ by Lemma 1, and (15) is satisfied.

In case (ii) the equation (15) is satisfied because of $\langle V_i, V_i\rangle = 1$.

In case (iii) we obtain by (8), (10), (13): $X(\lambda_i) - \langle X, V_i\rangle V_i(\lambda_i) = V_j(\lambda_i) - \tau_{ij}V_i(\lambda_i) = \eta_{ij} + k\tau_{ij} = 0$. This completes the proof of statement (a) of Theorem 1. ∎

II. Proof of (b). Let T be a distribution $T = \operatorname{span}\{V_1, \cdots, V_r\}$ and let $T^\perp$ be the orthogonal complement of T.

Lemma 4. *The distributions T and $T^\perp$ are involutive.*

Proof. By (7),

$$[V_i, V_j] = \nabla_{V_i}V_j - \nabla_{V_j}V_i = (\lambda_j + \lambda_i\tau_{ij})V_i - (\lambda_i + \lambda_j\tau_{ij})V_j. \tag{16}$$

Hence $[V_i, V_j] \in T$ and T is involutive.

Let $X, Y \in T^\perp$. Then $X\langle Y, V_i\rangle = 0$. Therefore by (7), $\langle\nabla_Y X, V_i\rangle + \lambda_i\langle X, Y\rangle = 0$. Analogously $\langle\nabla_X Y, V_i\rangle + \lambda_i\langle X, Y\rangle = 0$. It follows that $\langle[X, Y], V_i\rangle = \langle\nabla_X Y, V_i\rangle - \langle\nabla_Y X, V_i\rangle = 0$. Hence $[X, Y] \in T^\perp$. ∎

Let us define a linear differential form θ on M by the equations

$$\begin{aligned} &\theta(V_i) = \lambda_i \qquad (i = 1, \cdots, r);\\ &\theta(X) = 0 \qquad \text{if } X \in T^\perp. \end{aligned} \tag{17}$$

LEMMA 5. $d\theta = 0$.

PROOF. We have to prove that $d\theta(X, Y) = 0$ for every $X, Y \in \chi(M)$. It is enough to consider three cases: (1) $X = V_i$, $Y = V_j$; (2) $X = V_i$, $Y \in T^\perp$; (3) $X, Y \in T^\perp$. In case (1) we obtain by (8), (16) and (ii) of Lemma 2:

$$d\theta(V_i, V_j) = \tfrac{1}{2}[V_i(\lambda_j) - V_j(\lambda_i) - (\lambda_j + \lambda_i\tau_{ij})\lambda_i + (\lambda_i + \lambda_j\tau_{ij})\lambda_j] = 0.$$

In cases (2) and (3) $d\theta(X, Y) = 0$ by Lemmas 1 and 4. ∎

Let $p \in M$. Lemma 5 shows that there exist a coordinate neighbourhood U of p and a function ψ on U such that $\theta|_U = d\psi$. It is clear that on U

$$(18)\qquad \begin{aligned} V_i(\psi) &= \lambda_i \qquad (i = 1, \cdots, r). \\ X(\psi) &= 0 \qquad \text{for } X \in T^\perp. \end{aligned}$$

Since, by Lemma 4, the distributions T and $T^\perp$ are involutive, we can choose U to be diffeomorphic to $U_0 \times U_1$, where U_0 and U_1 are the slices of the distributions T and $T^\perp$ through p on U. Let $(x^1, \cdots, x^n)$ be a local coordinate system on U with the origin at p, such that $(\partial/\partial x^1, \cdots, \partial/\partial x^r)$ and $(\partial/\partial x^{r+1}, \cdots, \partial/\partial x^n)$ form local bases for T and $T^\perp$ respectively. Note that U_0 is defined by the equations $x^\alpha = 0$ and U_1 is defined by the equations $x^i = 0$ (here and in the following i, j, k, m take the values $1, \cdots, r$; α, β, γ take the values $r+1, \cdots, n$, and a, b, c take the values $1, \cdots, n$). It follows from (18) and Lemma 1 that τ_{ij}, λ_i and ψ do not depend on x^α, and therefore may be regarded as functions on U_0.

Let $V_i = V^a_{(i)}\partial/\partial x^a$, where $V^\alpha_{(i)} = 0$. Equation (1) can be rewritten in a coordinate form

$$(19)\qquad V^a_{(i),b} = \lambda_i(\delta^a_b - V^a_{(i)}V_{(i)b}),$$

where comma denotes the covariant derivative. Taking in (19) $a = \alpha$, $b = j$ we obtain $V^\alpha_{(i),j} = 0$. This implies $\Gamma^\alpha_{jk} = 0$ and therefore $\partial g_{ij}/\partial x^\alpha = 0$. Hence the g_{ij} do not depend on x^α and may be regarded as functions on U_0. Taking in (19) $a = \alpha$, $b = \beta$ we obtain $\Gamma^\alpha_{\beta k}V^k_{(i)} = \lambda_i\delta^\alpha_\beta$. Let $U^{(i)}_k$ be the inverse matrix of $V^k_{(i)}$. Then $\Gamma^\alpha_{\beta k} = \delta^\alpha_\beta \Sigma_i \lambda_i U^{(i)}_k$. This implies

$$(20)\qquad \partial g_{\alpha\beta}/\partial x^k = 2g_{\alpha\beta}\sum_i \lambda_i U^{(i)}_k.$$

Since $\partial/\partial x^k = \Sigma_i U^{(i)}_k V_i$, we obtain $\partial\psi/\partial x^k = \Sigma_i U^{(i)}_k V_i(\psi)$. Therefore by (18),

$$(21)\qquad \partial\psi/\partial x^k = \sum_i \lambda_i U^{(i)}_k.$$

Let us denote $\bar{g}_{\alpha\beta} = e^{-2\psi}g_{\alpha\beta}$. It follows from (20), (21) that $\partial\bar{g}_{\alpha\beta}/\partial x^k = 0$. Hence the $\bar{g}_{\alpha\beta}$ do not depend on x^i and may be regarded as functions on U_1. We see that

$$g_{ab}dx^a dx^b = g_{ij}dx^i dx^j + e^{2\psi}\bar{g}_{\alpha\beta}dx^\alpha dx^\beta,$$

where $g_{ij}dx^i dx^j$ is a metric on U_0, $\bar{g}_{\alpha\beta}dx^\alpha dx^\beta$ is a metric on U_1 and $e^{2\psi}$ is a positive valued function on U_0. Therefore U is a warped product $U_0 \times_{e^{2\psi}} U_1$.

Taking in (19) $a = j$, $b = \alpha$ we obtain

$$\partial V^j_{(i)}/\partial x^\alpha = V^j_{(i),\alpha} - \Gamma^j_{\alpha k}V^k_{(i)} = -\Gamma^j_{\alpha k}V^k_{(i)}.$$

But $\Gamma^j_{\alpha k} = \frac{1}{2}g^{jm}\partial g_{km}/\partial x^\alpha = 0$. Therefore $\partial V^j_{(i)}/\partial x^\alpha = 0$. Hence the vector fields V_i on $U = U_0 \times U_1$ may be regarded as fields induced by vector fields $W_i = V_i\big|_{U_0}$ defined on U_0. Moreover, a direct computation shows that

$$W^j_{(i)/k} = \lambda_i(\delta^j_k - W^j_{(i)}W_{(i)k}),$$

where / denotes the covariant derivative on U_0 with respect to the metric $g_{ij}dx^i dx^j$. In the case $r \geqq 3$, this means that the W_i are LSVF's of the second order. If $r = 2$ and if V_2 is a LSVF of the second order, then $V_1(\lambda_2) = \tau_{12}V_2(\lambda_2)$. The latter is equivalent to the equation $W_1(\lambda_2) = \tau_{12}W_2(\lambda_2)$. As in the proof of part (a) of Theorem 1, it follows that W_1 is a LSVF of the second order.

To complete the proof of part (b) of Theorem 1 we only have to show that $U = U_0 \times_{e^{2\psi}} U_1$ is a k-decomposition. Since there exist r LSVF's W_i on the r-dimensional manifold U_0, it follows from (9) that for $r \geqq 3$, U_0 is of constant curvature. The same is valid for $r = 2$, as can be readily deduced from the following lemma.

LEMMA 6. *Let M be a 2-dimensional Riemannian manifold and let V_1 and V_2 be two linearly independent unit vector fields on M satisfying the equations*

$$\nabla_X V_i = \lambda_i(X - \langle X, V_i\rangle V_i) \qquad (i = 1, 2). \tag{22}$$

$$X(\lambda_i) = \langle X, V_i\rangle V_i(\lambda_i) \qquad (i = 1, 2), \tag{23}$$

where X is an arbitrary vector field on M, and λ_1, λ_2 are functions on M. Then M is a manifold of constant curvature.

PROOF. The condition $n > 2$ was not used in the proof of equation (10) of Lemma 2. Thus, from (10) and (23):

$$V_i(\lambda_i) = \lambda_i^2 - k, \qquad V_j(\lambda_i) = \tau_{ij}(\lambda_i^2 - k).$$

Now computing $(V_1V_2 - V_2V_1)(\lambda_i)$ and using $V_1(\tau_{12}) = V_1\langle V_1, V_2\rangle = \lambda_2(1-\tau_{12}^2)$, $V_2(\tau_{12}) = \lambda_1(1-\tau_{12}^2)$, $[V_1, V_2] = \nabla_{V_1}V_2 - \nabla_{V_2}V_1 = (\lambda_2 + \lambda_1\tau_{12})V_1 - (\lambda_1 + \lambda_2\tau_{12})V_2$, we obtain $V_i(k) = 0$. Therefore $k = \text{const}$. ■

Returning to the proof of the theorem, equation (21) may be rewritten as $W_i(\psi) = \mu_i$, where $\mu_i = \lambda_i|_{U_0}$. Now

$$\begin{aligned}\langle \nabla_{W_i}\operatorname{grad}\psi + kW_i + W_i(\psi)\operatorname{grad}\psi, W_j\rangle \\ &= W_i(W_j(\psi)) - \nabla_{W_i}W_j(\psi) + k\tau_{ij} + W_i(\psi)W_j(\psi) \\ &= W_i(\mu_j) - \mu_j(W_i(\psi) - \tau_{ij}W_j(\psi)) + k\tau_{ij} + \mu_i\mu_j \\ &= W_i(\mu_j) + \mu_j^2\tau_{ij} + k\tau_{ij} = \eta_{ij} + k\tau_{ij} \\ &= 0,\end{aligned}$$

by Lemma 3. Therefore $\nabla_{W_i}\operatorname{grad}\psi = -kW_i - W_i(\psi)\operatorname{grad}\psi$. Because of part (c) of Proposition 2, we have that $U = U_0 \times_{e^{2\psi}} U_1$ is a k-decomposition.

This completes the proof of statement (b) of Theorem 1. ■

III. Proof of (c). This is completely analogous to that of (b), and is omitted.

IV. Proof of (d). Let (W_i, μ_i) $(i = 1, \cdots, r)$ be LSVF's of the second order on a manifold M_0 of constant curvature. Let us define a linear differential form θ by the equations $\theta(W_i) = \mu_i$. By Lemma 5, $d\theta = 0$. Since M_0 is simply connected, there exists a function ψ on M_0 satisfying $\theta = d\psi$. It follows that $W_i(\psi) = \mu_i$. The rest of the proof is analogous to that of part (b). ■

V. Proof of (e). By (b) of the present theorem, locally M has a k-decomposition $U_0 \times_{e^{2\psi}} U_1$, where $\dim U_1 = 1$. Therefore locally M is $U_0 \times_{e^{2\psi}} R^1$. By the definition of k-decomposition this means that M is a manifold of constant curvature. ■

This completes the proof of Theorem 1. ■

The following result is of global nature.

Theorem 2. *Let M be a n-dimensional $(n \geqq 4)$, connected, simply connected, and complete Riemannian manifold. If M admits r $(r \geqq 2)$ linearly independent LSVF's of the second order, then M is a warped product $M = M_0 \times_f M_1$, where M_0 is either R^r or simply connected hyperbolic space H^r. In particular*: (i) *M is noncompact*; (ii) *if M admits $(n-1)$ LSVF's of the second order, then $M = R^n$ or $M = H^n$.*

PROOF. By (b) of Theorem 1, M is locally a warped product. Since M is complete and simply connected it is also globally a warped product. This fact is an immediate consequence of theorem 1 in [2]. Thus, $M = M_0 \times_f M_1$, where $\dim M_0 = r$ and M_0 is connected, simply connected, complete (see [1]), and admits r LSVF's of the second order.

If $r = 2$ then M_0 has constant curvature k by Lemma 6. The possibility $k > 0$ is ruled out since the sphere S^2 does not admit a nonvanishing vector field. Therefore $k \leqq 0$, and M_0 is either R^2 or H^2.

If $r \geqq 3$ then M_0 has a constant curvature k by (9). In this case let us select two fields (V_1, λ_1) and (V_2, λ_2) from a given set of r LSVF's on M. These fields define a warped decomposition $M = N_0 \times_\varphi N_1$, where N_0 is 2-dimensional, simply connected, complete, and admitting two LSFV's of the second order. Once more by Lemma 6, N has a constant non-positive curvature. By (a) of Proposition 2, this curvature is equal to the curvature k of M_0. Therefore M_0 is either R^r or H^r.

■

4. The case $n = 3$

Let M be a 3-dimensional Riemannian manifold and let (V, λ, β) be a LSVF of the first order on M, i.e., V is a unit vector field satisfying the equation

$$\nabla_X V = \lambda (X - \langle X, V\rangle V) + \beta (V \times X), \tag{24}$$

where λ and β are functions on M and $X \in \chi(M)$ is an arbitrary vector field on M.

LEMMA 7. *For every* $X, Y \in \chi(M)$

$$\langle R(X, V)V, Y\rangle = k(\langle X, Y\rangle - \langle X, V\rangle\langle Y, V\rangle), \tag{25}$$

where

$$k = -[V(\lambda) + \lambda^2 - \beta^2]. \tag{26}$$

PROOF.

$$\langle R(X, V)V, Y\rangle = -\langle \nabla_V \nabla_X V - \nabla_X \nabla_V V - \nabla_{[V,X]} V, Y\rangle =$$

$$\langle \nabla_V(\nabla_X V) - \lambda([V, X] - \langle [V, X], V\rangle V) + \beta(V \times [V, X]), Y\rangle.$$

Using (24) and substituting

$$[V, X] = \nabla_V X - \nabla_X V = \nabla_V X - \lambda (X - \langle X, V\rangle V) + \beta (V \times X),$$

we obtain

$$\langle R(X,V)V,Y\rangle = -[V(\lambda)+\lambda^2-\beta^2](\langle X,Y\rangle - \langle X,V\rangle\langle Y,V\rangle)$$
$$-(V(\beta)+2\lambda\beta)\langle (V\times X),Y\rangle.$$

Since $\langle R(X,V)V,Y\rangle$ is symmetric on X and Y, and $\langle (V\times X),Y\rangle$ is antisymmetric, we get (25). ■

Let S be the Ricci tensor corresponding to the curvature tensor R, let μ_1,μ_2,μ_3 be eigenvalues of S, and let e_1,e_2,e_3 be linearly independent unit eigenvectors which correspond to μ_1,μ_2,μ_3.

THEOREM 3. *Let (V,λ,β) be a LSVF of the first order on M. Then*

(a) *If μ_1,μ_2,μ_3 are pairwise different $(\mu_1<\mu_3<\mu_2)$, then V is one of the following four vector fields:*

$$V=\pm\left(\frac{\mu_3-\mu_1}{\mu_2-\mu_1}\right)^{1/2}e_1\pm\left(\frac{\mu_2-\mu_3}{\mu_2-\mu_1}\right)^{1/2}e_2. \tag{27}$$

(b) *If $\mu_2=\mu_3$, but $\mu_1\neq\mu_2,\mu_3$, then*

$$V=\pm e_1. \tag{28}$$

PROOF. The proof is purely algebraic, using only equation (25). Let p be some fixed point on M. It is sufficient to prove equalities (27), (28) at this point. Therefore we can consider μ_1,μ_2,μ_3,k, and the components of the tensors R and S, as real numbers, and V,X,Y as vectors in R^3.

I. PROOF OF (a). Let $X_2,X_3\in R^3$, $\langle X_2,V\rangle=0$, $\langle X_3,V\rangle=0$, $\langle X_2,X_3\rangle=0$, $\|X_2\|=\|X_3\|=1$. Then by (25), $S(X_2,X_3)=\langle R(X_2,V)X_3,V\rangle=0$;

$$S(X_2,X_2)-S(X_3,X_3)=\langle R(X_2,V)X_2,V\rangle+\langle R(X_2,X_3)X_2,X_3\rangle$$
$$-\langle R(X_3,V)X_3,V\rangle-\langle R(X_3,X_2)X_3,X_2\rangle$$
$$=k-k=0.$$

It follows that for the frame $\{X_1=V,X_2,X_3\}$, $S_{23}=0$, $S_{22}=S_{33}$. The characteristic equation for S takes the form

$$f(\mu)=(\mu-S_{33})[(\mu-S_{33})(\mu-S_{11})-(S_{12}^2+S_{13}^2)]=0.$$

One of the roots is $\mu=\mu_3=S_{33}$. The other roots μ_1,μ_2 can be found from the equation

$$(29) \qquad (\mu - \mu_3)(\mu - S_{11}) - (S_{12}^2 + S_{13}^2) = 0.$$

It is easily verified that $\mu_1 < \mu_3 < \mu_2$ so that our labeling of the roots is consistent. Let us find the eigenvector $e_3 = xV + yX_2 + zX_3$ corresponding to $\mu = \mu_3$:

$$(30) \qquad \begin{aligned} (S_{11} - \mu_3)x + S_{12}y + S_{13}z &= 0, \\ S_{12}x &= 0, \\ S_{13}x &= 0. \end{aligned}$$

If $x \neq 0$ then $S_{12} = S_{13} = 0$, and $S_{11} = \mu_3$, so that the eigenvalues are equal. Since this is not the case, we get $x = 0$. Therefore $\langle e_3, V \rangle = 0$. Let us now take $X_1 = V$, $X_3 = e_3$. Then $S_{13} = 0$. Suppose $e_1 = x_1X_1 + y_1X_2$, $e_2 = x_2X_1 + y_2X_2$. Then

$$(S_{11} - \mu_2)x_2 + S_{12}y_2 = 0,$$

$$S_{12}x_2 + (\mu_3 - \mu_2)y_2 = 0.$$

We see that the vector $S_{12}X_1 + (\mu_3 - \mu_2)X_2$ is perpendicular to e_2. Hence it is collinear with e_1 and we have

$$e_1 = \pm [S_{12}^2 + (\mu_3 - \mu_2)^2]^{-\frac{1}{2}}(S_{12}V + (\mu_3 - \mu_2)X_2).$$

Let us denote $\cos \tau = \langle e_1, V \rangle$. Then $V = \cos \tau e_1 + \sin \tau e_2$. The proof will be completed by showing that $\cos^2 \tau = (\mu_3 - \mu_1)/(\mu_2 - \mu_1)$. We have

$$\cos^2 \tau - \frac{\mu_3 - \mu_1}{\mu_2 - \mu_1} = \frac{S_{12}^2}{S_{12}^2 - (\mu_3 - \mu_2)^2} - \frac{\mu_3 - \mu_1}{\mu_2 - \mu_1}$$

$$= \frac{\mu_2 - \mu_3}{[S_{12}^2 - (\mu_3 - \mu_2)^2](\mu_2 - \mu_1)}[S_{12}^2 + \mu_3^2 - \mu_3(\mu_1 + \mu_2) + \mu_1\mu_2].$$

From (30) we get $\mu_1 + \mu_2 = \mu_3 + S_{11}$, $\mu_1\mu_2 = \mu_3S_{11} - S_{12}^2$. Substituting into the last formula, we obtain $\cos^2 \tau - (\mu_3 - \mu_1)/(\mu_2 - \mu_1) = 0$. This completes the proof of part (a).

II. Proof of (b). In this case $\mu_2 = \mu_3$, $\mu_1 \neq \mu_2, \mu_3$. Let $\langle X_2, V \rangle = 0$, $\langle X_3, V \rangle = 0$, $\|X_2\| = \|X_3\| = 1$. As in the proof of (a) we obtain for the frame $\{X_1 = V, X_2, X_3\}$, $S_{23} = 0$, $S_{22} = S_{33}$. The characteristic equation for S takes the form

$$f(\mu) = (\mu - S_{33})[(\mu - S_{33})(\mu - S_{11}) - (S_{12}^2 + S_{13}^2)] = 0.$$

The second factor has two equal roots only if $S_{33} = S_{11}$, $S_{12} = S_{13} = 0$, and we obtain $\mu_1 = \mu_2 = \mu_3$. This is not the case. Hence $\mu = S_{33}$ must be a root of the second factor, and we obtain $S_{12}^2 + S_{13}^2 = 0$, i.e., $S_{12} = S_{13} = 0$. The Ricci tensor takes a diagonal form and therefore V is an eigenvector: $V = \pm e_1$. ■

If (V, λ, β) is a LSVF of the first order, so is the vector $(-V, \lambda, -\beta)$. If we do not distinguish between such two LSVF's, we obtain as an immediate consequence of Theorem 3 the following theorem.

THEOREM 4. *Let* $\dim M = 3$.

(a) *If the eigenvalues of the Ricci tensor are pairwise different, then M admits at most two different LSVF's of the first order.*

(b) *If two eigenvalues of the Ricci tensor are equal but the third is distinct, then M admits at most one LSVF of the first order.*

(c) *If M admits three different (but not necessarily linearly independent) LSVF's of the first order, then M is a manifold of constant curvature.*

THEOREM 5. *Let* $\dim M = 3$. *If V is a LSVF of the second order on M, then at every point of M, V is an eigenvector of the Ricci tensor S.*

PROOF. By Proposition 1, V satisfies one of the conditions

(i) $$X(\lambda) = \langle X, V\rangle V(\lambda),$$

(31) $$\beta = 0$$

for every $X \in \chi(M)$,

(32) (ii) $$\lambda = 0; \qquad \beta = \text{const} \neq 0.$$

To prove the theorem we use formula (24), page 142 of [6].

If V satisfies equation (31) this formula reads

(33) $$S(X, Y) = [\tfrac{1}{2}\tilde{S} - V(\lambda) - \lambda^2]\langle X, Y\rangle - [\tfrac{1}{2}\tilde{S} - 3V(\lambda) - 3\lambda^2]\langle V, X\rangle\langle V, Y\rangle$$

where $\tilde{S}$ is the scalar curvature.

If V is not an eigenvector of S, there exist two orthonormal eigenvectors e_1 and e_2 of S such that $\langle V, e_1\rangle \neq 0$, $\langle V, e_2\rangle \neq 0$. Taking $X = e_1$ and $Y = e_2$ in (33), we obtain $\frac{1}{2}\tilde{S} - 3V(\lambda) - 3\lambda^2 = 0$, and therefore $S(X, Y) = \frac{1}{3}\tilde{S}\langle X, Y\rangle$. It follows that M is of constant curvature and V is an eigenvector, contradicting our assumption.

If V satisfies equation (32), equation (24), page 142 of [6] reads

$$S(X, Y) = (\tfrac{1}{2}\tilde{S} + \beta^2)\langle X, Y\rangle - (\tfrac{1}{2}\tilde{S} + 3\beta^2)\langle V, X\rangle\langle V, Y\rangle.$$

As in the previous paragraph we find that V is an eigenvector of S. ■

As an immediate consequence of Theorems 3 and 5 we obtain

THEOREM 6. *Let* $\dim M = 3$. *If* M *admits two linearly independent LSVF's one of which is of the second order, then* M *is a manifold of constant curvature.*

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